4.3 Poincaré Disk as a Hilbert Plane.

Proposition 4.3.1

Let $f \in Aut(\mathbb{D})$. If ℓ is a P-line, then $f(\ell)$ is also a P-line.

proof.

If ℓ is a P-line, then $\ell = \gamma \cap \mathbb{D}$ where γ is a straight line passing through the origin or a circle perpendicular to $\mathcal{T}$.

Then there exist a pair of points z and $\frac{1}{\bar z}$ on γ.

Let $f(z) = \lambda \frac{z-a}{\bar a z - 1}$ where $a, \lambda \in \mathbb{C}$, $|a|<1$, $|\lambda|=1$.

Firstly, by exercise 4.1.4, $f(\gamma)$ is a straight line or a circle on $\mathbb{C}$.

Furthermore, $\frac{1}{\overline{f(z)}} = \frac{1}{\bar\lambda \frac{\bar z - \bar a}{a\bar z - 1}} = \lambda \frac{a\bar z - 1}{\bar z - \bar a}$ and $f(\frac{1}{\bar z}) = \lambda \frac{\frac{1}{\bar z} - a}{\frac{\bar a}{\bar z} - 1} = \lambda \frac{a\bar z - 1}{\bar z - \bar a}$

Therefore, $f(\gamma)$ contains $f(z)$ and $f(\frac{1}{\bar z}) = \frac{1}{\overline{f(z)}}$ and so $f(\gamma)$ is a straight line passing through the origin or a circle perpendicular to $\mathcal{T}$.

Note that $f : \mathbb{D} \to \mathbb{D}$ is bijective, so $f(\ell) = f(\gamma) \cap \mathbb{D}$ which is a P-line.

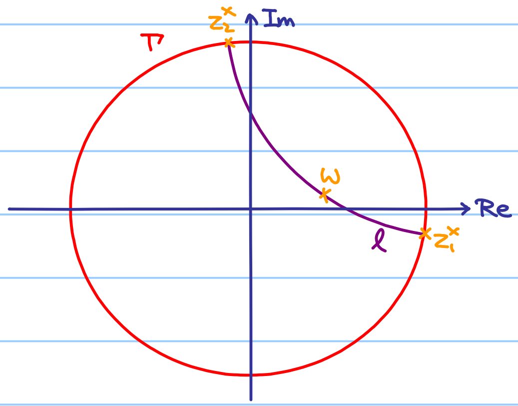
Also, if $|z|=1$, then $|f(z)|=1$.

Therefore, f maps the two boundary points of ℓ to those of $f(\ell)$.

Proposition 4.3.2

Let $f \in Aut(\mathbb{D})$. If ℓ is a P-line and $w \in \ell$, then there exists (exist exactly two) $f \in Aut(\mathbb{D})$ such that $f(w) = 0$ and $f(\ell) = \{x + i0 \in \mathbb{C} : -1 \le x \le 1\}$.

proof.



If $f \in Aut(\mathbb{D})$, then $f(z) = \lambda \frac{z-a}{\bar a z - 1}$

where $a, \lambda \in \mathbb{C}$, $|a|<1$, $|\lambda|=1$.

1) $f(w) = 0 \Leftrightarrow a = w$.

2) Let z_1^* and z_2^* be two boundary points of ℓ.

Since $|z_1^*| = |z_2^*| = 1$, by exercise 4.1.3, $\left|\frac{z_1^* - w}{\bar w z_1^* - 1}\right| = \left|\frac{z_2^* - w}{\bar w z_2^* - 1}\right| = 1$

If $\lambda_1 = \frac{z_1^* - w}{\bar w z_1^* - 1}$, $\lambda_2 = \frac{z_2^* - w}{\bar w z_2^* - 1}$.

Let $\lambda = \bar\lambda_1$ (or $\bar\lambda_2$), then $f(\lambda_1) = 1$ (or $f(\lambda_2) = 1$).

Furthermore, $f(z) = \frac{\bar z_1^* - \bar w}{w \bar z_1^* - 1} \frac{z - w}{\bar w z - 1}$ (or $f(z) = \frac{\bar z_2^* - \bar w}{w \bar z_2^* - 1} \frac{z - w}{\bar w z - 1}$) maps a P-line to a P-line.

However, the only P-line passing through the origin with 1 as endpoint is $\{x + i0 \in \mathbb{C} : -1 \le x \le 1\}$,

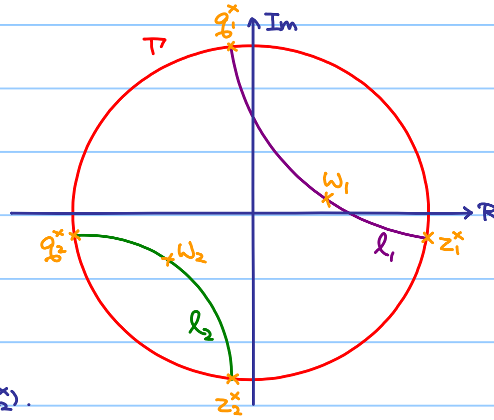
so $f(\ell) = \{x + i0 \in \mathbb{C} : -1 \le x \le 1\}$.

Corollary 4.3.1

Let ℓ_1 and ℓ_2 are P-lines on $\mathbb{D}$ and let $w_1 \in \ell_1$ and $w_2 \in \ell$.

Then there exists (exist exactly two) $f \in \mathrm{Aut}(\mathbb{D})$ such that $f(w_1) = w_2$ and $f(\ell_1) = \ell_2$.

proof:



Let $f_1, f_2 \in \mathrm{Aut}(\mathbb{D})$ such that $f_1(w_1) = f_2(w_2) = 0$ and

$f_1(z_1^*)$ (or $f_1(q_1^*)$) $= 1$ and $f_2(z_2^*)$ (or $f_2(q_2^*)$) $= 1$.

Then, the result follows by taking

$f(z) = (f_2^{-1} \circ f_1)(z) = f_2^{-1}(f_1(z))$.

Remark: f is uniquely determined if we know $f(z_1^*)$ or $f(z_2^*)$.

We are going to define the notion of point, line, betweenness, congruence for line segments and angles of the Poincaré Disk model.

Definition 4.3.1

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

If $p \in \mathbb{D}$, then p is said to be a point (or a P-point) on $\mathbb{D}$.

Let γ be a straight line on $\mathbb{C}$ passing through the origin or a circle perpendicular to the unit circle $\mathbb{T}$, then $\ell = \gamma \cap \mathbb{D}$ is said to be a line (or a P-line) on $\mathbb{D}$.

Proposition 4.3.3

Poincaré Disk model satisfies axioms of incidences (I1)-(I3)

Definition 4.3.2

Let ℓ be any P-line in $\mathbb{D}$. Then, ℓ is given by $\gamma : (c_0, c_1) \to \mathbb{D}$.

Let $A = \gamma(t_1)$, $B = \gamma(t_2)$, $C = \gamma(t_3)$ be distinct points on ℓ.

We define $A * B * C$ if $t_1 < t_2 < t_3$ or $t_1 > t_2 > t_3$.

Proposition 4.3.2

Poincaré Disk model satisfies axioms of betweenness (B1)-(B4)

Exercises 4.3.1

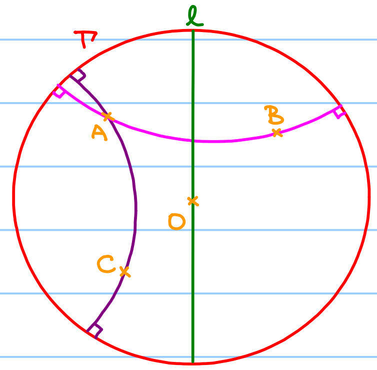
Let ℓ be a P-line which is the diameter (without endpoints) of Γ.
Then ℓ separates $\mathbb{D}$ into two disjoint subsets. Show that

(i) A, B lie on different subsets

$\Leftrightarrow$ the P-line segment AB interects ℓ exactly once.

(ii) A, C lie on the same subsets

$\Leftrightarrow$ the P-line passing through AC does not interects ℓ.

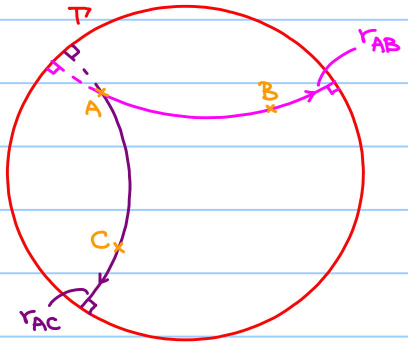


Corollary 4.3.2

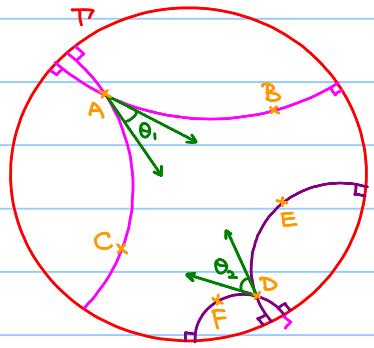
Let r_1 and r_2 are P-rays on $\mathbb{D}$ originating at w_1 and w_2 respectively.
Then there exists unique $f \in Aut(\mathbb{D})$ such that $f(w_1) = w_2$ and $f(r_1) = r_2$.

Definition 4.3.2

Let AB and CD be P-line segments. We define $AB \cong CD$ if $d(A,B) = d(C,D)$.
Let $\angle BAC$ and $\angle EDF$ be P-angles. We define $\angle BAC \cong \angle EDF$ if $\theta_1 = \theta_2$.



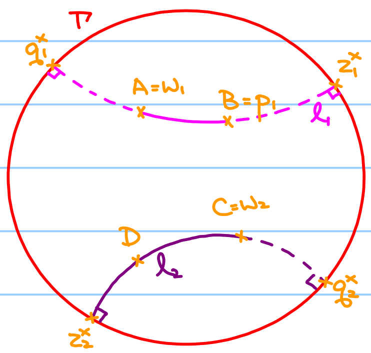
$\angle BAC = r_{AB} \cup r_{AC}$



Proposition 4.3.3

Poincaré Disk model satisfies axioms of congruence for line segments and angles (C1)-(C6).

(C1) :



There exists unique $f \in Aut(\mathbb{D})$ such that
$f(w_1) = w_2$ and $f(z_1^*) = z_2^*$.
Then, let $D = f(p_1)$, and
we have $d(A,B) = d(w_1, p_1) = d(f(w_1), f(p_1)) = d(C,D)$

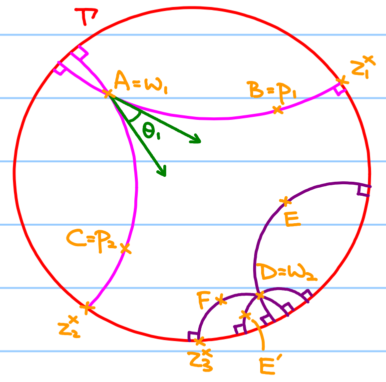
(C4) :

There exists unique $f \in Aut(\mathbb{D})$ such that

$f(w_1) = w_2$ and $f(z_1^*) = z_3^*$ (or $f(z_2^*) = z_3^*$)

Let $E = f(p_2)$ (or $E = f(p_1)$)

Then $\angle BAC \cong \angle EDF$.



Theorem 4.3.1

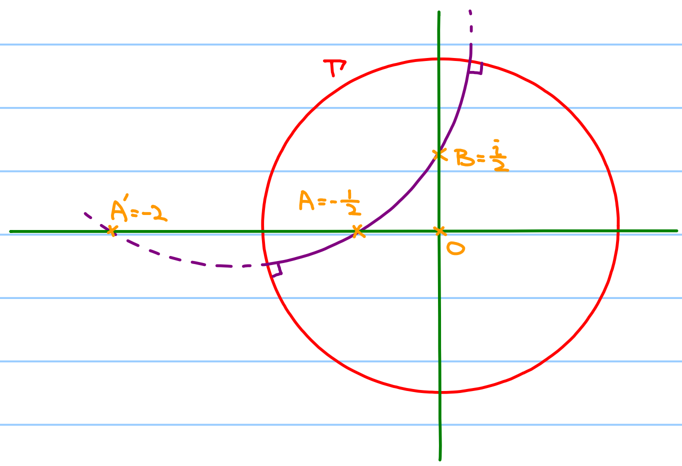
Poincaré Disk model is a Hilbert plane.

Example 4.3.1

a) Find the equation of P-line passing through $A = -\frac{1}{2}$ and $B = \frac{i}{2}$

It is the intersection of $\mathbb{D}$ and the circle passing through $A = -\frac{1}{2}$, $B = \frac{i}{2}$ and $A' = -2$.

$$(x+\tfrac{5}{4})^2 + (y-\tfrac{5}{4})^2 = \tfrac{17}{8}, \quad \text{for } x^2+y^2<1.$$



b) Find the P-angle $\angle OAB$.

$$(x+\tfrac{5}{4})^2 + (y-\tfrac{5}{4})^2 = \tfrac{17}{8}$$

Differentiate both sides with respect to x:

$$2(x+\tfrac{5}{4}) + 2(y-\tfrac{5}{4})\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4x+5}{4y-5}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(-\frac{1}{2},0)} = \frac{3}{7}$$

P-angle $\angle OAB = \tan^{-1}\frac{3}{7} \approx 23.1^\circ < 45^\circ$

Remark: Sum of interior P-angles of P-triangle $\Delta OAB < 180^\circ$.

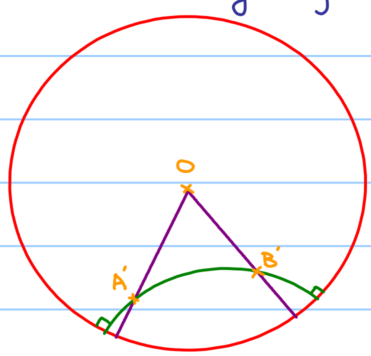
For a general P-triangle ΔABC, by choosing a suitable $f \in Aut(\mathbb{D})$, the images of A, B, C under f are A', B' and O', then we have

P-angle $\angle ACB$ = P-angle $\angle A'O'B' = \angle A'OB'$

P-angle $\angle BAC$ = P-angle $\angle B'A'O < \angle B'A'O$

P-angle $\angle CBA$ = P-angle $\angle OB'A' < \angle OB'A'$

$\therefore$ Sum of interior P-angles of P-triangle $\Delta OAB < 180^\circ$.



Let $A, C \in \mathbb{D}$ and let γ be the circle with center C and radius CA.

What does γ look like ?

Let $B \in \gamma$. By choosing a suitable $f \in Aut(\mathbb{D})$, the images of A, B, C under f are A', B' and 0, then we have $d(B', 0) = d(B, C) = d(A, C) = d(A', 0)$.

Therefore $f(\gamma)$ is just the ordinary circle with center 0 and radius OA' on $\mathbb{D}$

However $\gamma = f^{-1}(f(\gamma))$ and $f^{-1} \in Aut(\mathbb{D})$ which maps a circle on $\mathbb{D}$ to another circle on $\mathbb{D}$.

As a result, we have the following proposition:

Proposition 4.3.4

γ is a P-circle on $\mathbb{D}$ if and only if γ is an ordinary circle on $\mathbb{D}$.

Example 4.3.2

Let $C = \frac{1}{2}$, $A = \frac{i}{2}$.

a) Find $d(A, C)$.

b) Find the equation of the circle γ center C and radius CA.

Let $f(z) = \dfrac{z - \frac{1}{2}}{\frac{1}{2}z - 1} = \dfrac{2z-1}{z-2}$.

a) $f(\frac{1}{2}) = 0$ and $f(\frac{i}{2}) = \frac{10}{17} - \frac{6}{17}i$, $|f(\frac{i}{2})| = \frac{2}{17}\sqrt{34}$.

$$d(A,C) = d(\tfrac{i}{2}, \tfrac{1}{2}) = d(f(\tfrac{i}{2}), f(\tfrac{1}{2})) = d(\tfrac{10}{17} - \tfrac{6}{17}i, 0) = \ln \frac{1 + |f(\frac{i}{2})|}{1 - |f(\frac{i}{2})|} = \ln \frac{17 + 2\sqrt{34}}{17 - 2\sqrt{34}}$$

b) Equation of $f(\gamma)$: $x^2 + y^2 = |f(\frac{i}{2})|^2 = \frac{8}{17}$

Let $z = x + iy$, $w = u + iv$ and $z = f(w)$.

$$x + iy = \frac{2(u+iv) - 1}{(u+iv) - 2}$$

$$= \frac{(2u-1) + i(2v)}{(u-2) + iv}$$

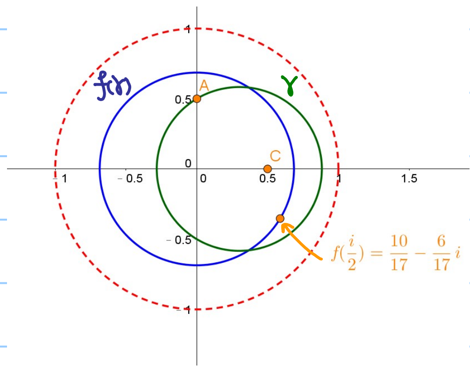
$$= \frac{2u^2 + 2v^2 - 5u + 2}{(u-2)^2 + v^2} + \frac{-3v}{(u-2)^2 + v^2} i$$

$$x^2 + y^2 = \frac{8}{17} \Rightarrow 17(2u^2 + 2v^2 - 5u + 2)^2 + 17(-3v)^2 = 8\left[(u-2)^2 + v^2\right]^2$$

$$\vdots$$

$$60\left[(u-2)^2 + v^2\right]\left[(u - \tfrac{3}{10})^2 + v^2 - \tfrac{34}{100}\right] = 0$$

$$(u - \tfrac{3}{10})^2 + v^2 = \frac{\sqrt{34}}{10}$$



4.4 Trigonometry

Goal: Study the relation between sides and angles of a P-triangle.

Definition 4.4.1

Hyperbolic cosine function : $\cosh(x) = \frac{e^x + e^{-x}}{2}$

Hyperbolic sine function : $\sinh(x) = \frac{e^x - e^{-x}}{2}$

Hyperbolic tangent function: $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Exercise 4.4.1

Prove the following identities:

1) $\frac{d}{dx}\sinh x = \cosh x$

2) $\frac{d}{dx}\cosh x = \sinh x$

3) $\cosh^2 x - \sinh^2 x = 1$

4) $\sinh 2x = 2\cosh x \sinh x$

5) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$

6) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

7) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

8) $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

9) Let $t = \tanh(\frac{x}{2})$

$\sinh x = \frac{2t}{1-t^2}$ $\quad \cosh x = \frac{1+t^2}{1-t^2}$ $\quad \tanh x = \frac{2t}{1+t^2}$

Let $y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ for all $x \in \mathbb{R}$.

Then $y = \frac{e^{2x} - 1}{e^{2x} + 1}$,

$e^{2x} = \frac{1+y}{1-y}$

$x = \frac{1}{2}\ln\frac{1+y}{1-y}$

$\therefore \tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$

Exercise 4.4.2

Show that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ and $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

With the above, let $z_1, z_2 \in \mathbb{D}$ and $f(z) = \frac{z - z_1}{\bar{z}_1 z - 1} \in \mathrm{Aut}(\mathbb{D})$. Then,

$$d(z_1, z_2) = d(f(z_1), f(z_2)) = d(0, f(z_2)) = \ln \frac{1+|f(z_2)|}{1-|f(z_2)|} = 2\tanh^{-1}\left|\frac{z_2 - z_1}{\bar{z}_1 z_2 - 1}\right|$$

Theorem 4.4.1 (Cosine Rule for Hyperbolic Triangle)

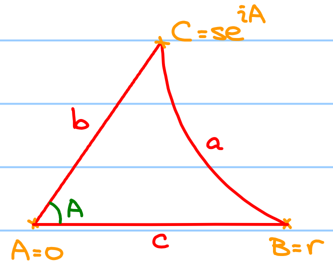
$$\sinh(b)\sinh(c)\cos A = \cosh(b)\cosh(c) - \cosh(a)$$

proof:

Only need to consider the following special case:

$s = \tanh(\frac{b}{2})$

$r = \tanh(\frac{c}{2})$



$$\tanh(\tfrac{a}{2}) = \left|\frac{se^{iA} - r}{rse^{iA} - 1}\right|$$

$$\tanh^2(\tfrac{a}{2}) = \frac{se^{iA} - r}{rse^{iA} - 1} \cdot \frac{se^{-iA} - r}{rse^{-iA} - 1} = \frac{r^2 + s^2 - 2rs\cos A}{r^2 s^2 + 1 - 2rs\cos A}$$

$$(r^2s^2+1)\tanh^2(\tfrac{a}{2}) - 2rs\tanh^2(\tfrac{a}{2})\cos A = r^2 + s^2 - 2rs\cos A$$

$$2rs(1-\tanh^2(\tfrac{a}{2}))\cos A = r^2 + s^2 - (r^2s^2+1)\tanh^2(\tfrac{a}{2})$$

$$4rs(1-\tanh^2(\tfrac{a}{2}))\cos A = 2r^2 + 2s^2 - 2(r^2s^2+1)\tanh^2(\tfrac{a}{2})$$

$$= (r^2s^2 + r^2 + s^2 + 1)(1-\tanh^2(\tfrac{a}{2})) - (r^2s^2 - r^2 - s^2 + 1)(1+\tanh^2(\tfrac{a}{2}))$$

$$= (1+r^2)(1+s^2)(1-\tanh^2(\tfrac{a}{2})) - (1-r^2)(1-s^2)(1+\tanh^2(\tfrac{a}{2}))$$

$$\frac{2r}{1-r^2} \cdot \frac{2s}{1-s^2} \cdot \cos A = \frac{(1+r^2)(1+s^2)}{(1-r^2)(1-s^2)} - \frac{1+\tanh^2(\frac{a}{2})}{1-\tanh^2(\frac{a}{2})}$$

$$\sinh(b)\sinh(c)\cos A = \cosh(b)\cosh(c) - \cosh(a)$$

In particular, if $A = \frac{\pi}{2}$,

$\cosh(b)\cosh(c) = \cosh(a)$ (Pythagoras' Theorem for Hyperbolic Triangle)

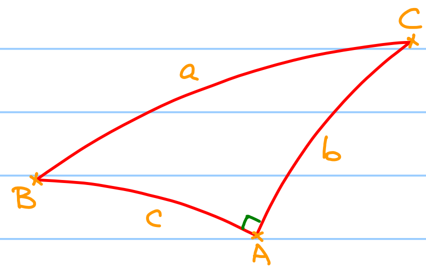
By cosine rule again,

$$\sinh(a)\sinh(c)\cos B = \cosh(a)\cosh(c) - \cosh(b)$$

$$= \cosh(a)\cosh(c) - \frac{\cosh(a)}{\cosh(c)} \quad (\because \cosh(b)\cosh(c) = \cosh(a))$$

$$= \frac{\cosh(a)\sinh^2(c)}{\cosh(c)}$$

$$\cos B = \frac{\tanh(c)}{\tanh(a)}$$



Similarly, $\cos C = \frac{\tanh(b)}{\tanh(a)}$

Theorem 4.4.2 (Cosine Rule for Hyperbolic Triangle)

$$\frac{\sin A}{\sinh(a)} = \frac{\sin B}{\sinh(b)} = \frac{\sin C}{\sinh(c)}$$

proof:

$$\sinh(b)\sinh(c)\cos A = \cosh(b)\cosh(c) - \cosh(a)$$

$$\sinh^2(b)\sinh^2(c)\cos^2 A = [\cosh(b)\cosh(c) - \cosh(a)]^2$$

$$\sinh^2(b)\sinh^2(c)(1-\sin^2 A) = \cosh^2(b)\cosh^2(c) + \cosh^2(a) - 2\cosh(a)\cosh(b)\cosh(c)$$

$$\begin{aligned}\sinh^2(b)\sinh^2(c)\sin^2 A &= \sinh^2(b)\sinh^2(c) - \cosh^2(b)\cosh^2(c) - \cosh^2(a) + 2\cosh(a)\cosh(b)\cosh(c)\\ &= (\cosh^2(b)-1)(\cosh^2(c)-1) - \cosh^2(b)\cosh^2(c) - \cosh^2(a) + 2\cosh(a)\cosh(b)\cosh(c)\\ &= 1 - \cosh^2(a) - \cosh^2(b) - \cosh^2(c) + 2\cosh(a)\cosh(b)\cosh(c) \quad \text{—— } (*)\end{aligned}$$

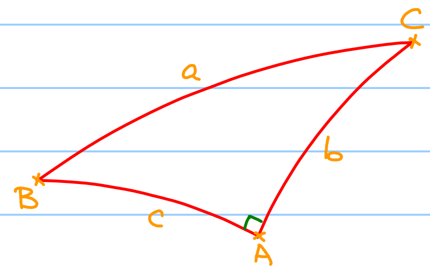
$$\frac{\sin^2 A}{\sinh^2(a)} = \frac{1 - \cosh^2(a) - \cosh^2(b) - \cosh^2(c) + 2\cosh(a)\cosh(b)\cosh(c)}{\sinh^2(a)\sinh^2(b)\sinh^2(c)} = \Delta$$

which is symmetric in a, b, c

$$\therefore \frac{\sin A}{\sinh(a)} = \frac{\sin B}{\sinh(b)} = \frac{\sin C}{\sinh(c)} = \sqrt{\Delta} \qquad \left(\text{Note: } \frac{\sin A}{\sinh(a)}, \frac{\sin B}{\sinh(b)}, \frac{\sin C}{\sinh(c)} > 0\right)$$

In particular, if $A = \frac{\pi}{2}$,

$$\sin B = \frac{\sinh(b)}{\sinh(a)} \qquad \sin C = \frac{\sinh(c)}{\sinh(a)}$$



Recall: $\cos B = \dfrac{\tanh(c)}{\tanh(a)}$

$$\begin{aligned}\therefore \tan B &= \frac{\sinh(b)\tanh(a)}{\sinh(a)\tanh(c)}\\ &= \frac{\sinh(b)\cosh(a)}{\tanh(c)}\\ &= \frac{\tanh(b)}{\sinh(c)} \qquad (\because \cosh(a) = \cosh(b)\cosh(c))\end{aligned}$$

Similarly, $\tan C = \dfrac{\tanh(c)}{\sinh(b)}$

Theorem 4.4.3 (Second Cosine Rule for Hyperbolic Triangle)

$\sin B \sin C \cosh(a) = \cos A + \cos B \cos C$

proof:

$\cos A = \dfrac{\cosh(b)\cosh(c) - \cosh(a)}{\sinh(b)\sinh(c)}$ $\quad \cos B = \dfrac{\cosh(a)\cosh(c) - \cosh(b)}{\sinh(a)\sinh(c)}$

$\cos C = \dfrac{\cosh(a)\cosh(b) - \cosh(c)}{\sinh(a)\sinh(b)}$

$$\cos A + \cos B \cos C = \frac{\cosh(b)\cosh(c) - \cosh(a)}{\sinh(b)\sinh(c)} + \frac{\cosh(a)\cosh(c) - \cosh(b)}{\sinh(a)\sinh(c)} \cdot \frac{\cosh(a)\cosh(b) - \cosh(c)}{\sinh(a)\sinh(b)}$$

$$= \frac{\sinh^2(a)[\cosh(b)\cosh(c) - \cosh(a)] + [\cosh(a)\cosh(c) - \cosh(b)][\cosh(a)\cosh(b) - \cosh(c)]}{\sinh^2(a)\sinh(b)\sinh(c)}$$

$$= \frac{2\cosh^2(a)\cosh(b)\cosh(c) - \sinh^2(a)\cosh(a) - \cosh(a)(\cosh^2(b) + \cosh^2(c))}{\sinh^2(a)\sinh(b)\sinh(c)}$$

$$= \frac{1 - \cosh^2(a) - \cosh^2(b) - \cosh^2(c) + 2\cosh(a)\cosh(b)\cosh(c)}{\sinh^2(a)\sinh(b)\sinh(c)} \cdot \cosh(a)$$

(Note: $(*) \Rightarrow \sinh^2(a)\sinh^2(b)\sin^2 C = 1 - \cosh^2(a) - \cosh^2(b) - \cosh^2(c) + 2\cosh(a)\cosh(b)\cosh(c)$)

$$= \frac{\sinh(b)\sin^2 C}{\sinh(c)} \cdot \cosh(a)$$

$$= \sin B \sin C \cosh(a)$$